

**2011
MATHCOUNTS STATE COMPETITION**

SPRINT ROUND

- 12 boy scouts are accompanied by 3 scout leaders. Each person needs 3 bottles of water per day and the trip is 1 day.
 $12 + 3 = 15$ people
 $15 \times 3 = 45$ bottles **Ans.**
- Cammie has pennies, nickels, dimes and quarters and we are asked to find the least number of coins that she can use to make 93 cents.
Let's start with quarters. 3 quarters equals 75¢. $93 - 75 = 18$ ¢.
We can use 1 dime. $18 - 10 = 8$
We can use 1 nickel. $8 - 5 = 3$
And we can use 3 pennies.
 $3 + 1 + 1 + 3 = 8$ **Ans.**
- We have 3 non-overlapping circles. One circle has a diameter of 8 inches. The other two circles have diameters of 6 and 2 inches, respectively. We are asked to find how much larger the area of the first circle is than the total area of the other 2 circles.
If the diameter of the first circle is 8, then its radius is 4. Its area is 16π .
The radii of the other two circles are 3 and 1, respectively. Their areas are 9π and π , respectively.
 $16\pi - (9\pi + \pi) = 6\pi$ **Ans.**
- Mike had 7 hits in 20 turns at bat in 5 baseball games. In the sixth game he had 5 hits in 5 turns at bat. We are asked to find the percent of Mike's turns at bat that resulted in hits.
The total number of turns at bat is $20 + 5 = 25$ turns
The total number of hits is $7 + 5 = 12$ hits
 $\frac{12}{25} = 48\%$ **Ans.**
- The first term of an arithmetic sequence is -37 and the second is -30 . We are asked to find the smallest positive term of the sequence.
There is a difference of 7 between the

two terms. If we look at the next 4 terms ($-23, -16, -9, -2$), we are adding 28 to the second term to get -2 as the sixth term.

$$-2 + 7 = 5 \text{ **Ans.**}$$

- Fraser fir tree saplings are 8 inches tall. Blue spruce tree saplings are 5 inches tall. Fraser firs grow at a constant rate of 12 inches per year and spruce trees at a constant rate of 14 inches per year. We are asked to find after how many years these trees will be the same height.
Let n = the number of years. Then,
 $8 + 12n = 5 + 14n$
 $3 = 2n$
 $n = \frac{3}{2}$ **Ans.**
- From the pie chart, $\frac{1}{3}$ of my \$30 allowance was spent on movies, $\frac{3}{10}$ on music, $\frac{1}{5}$ on ice cream and the rest on burgers.
 $\frac{1}{3} + \frac{3}{10} + \frac{1}{5} = \frac{10 + 9 + 6}{30} = \frac{25}{30}$
or \$25 out of \$30 were spent on items other than burgers.
 $30 - 25 = 5$ **Ans.**
- M and N are both perfect squares less than 100. If $M - N = 27$, then what is the value of $\sqrt{M} + \sqrt{N}$?
Let's list all the squares less than 100. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.
If we add 27 to each square we get 28, 31, 36, 43, 52, 63, 76, 91, 108 and 127.
36 is the only square. Therefore, $M = 9$ and $N = 36$
 $\sqrt{M} + \sqrt{N} = \sqrt{9} + \sqrt{36} = 3 + 6 = 9$ **Ans.**
- John made two 120-mile trips. The second trip took him one hour less than the first trip. Total time for both trips was 9 hours. We are asked to find his average rate for the second trip.
If the second trip took him one hour less

than the first and the total number of hours is 9, then the first trip was 5 hours long and the second was 4 hours long.

$$\frac{120}{4} = 30 \text{ Ans.}$$

10. We are asked to find the number of parallelograms in the diagram.



First the diagram itself contains 1 parallelogram. Each of the smallest items outlined in the parallelogram is itself a parallelogram.



There are 9 of those for a total of 10 so far.

There are also 6 "horizontal" parallelograms that contain 2 small parallelograms.

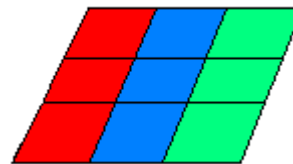
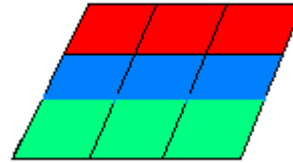


And there are 6 "vertical" parallelograms that contains 2 small parallelograms.



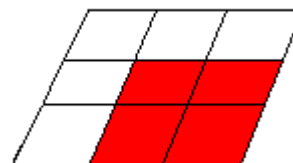
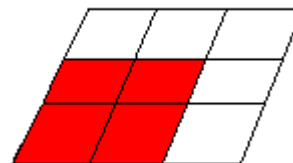
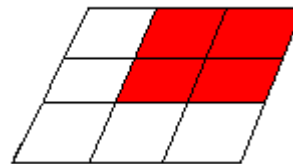
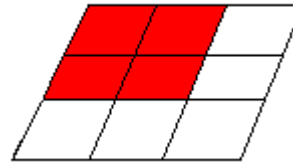
10 + 12 = 22 so far.

We can also make parallelograms from 3 small parallelograms. There are 3 "horizontal" ones and 3 "vertical" ones.



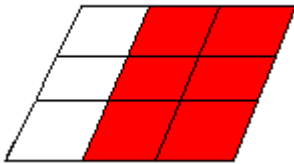
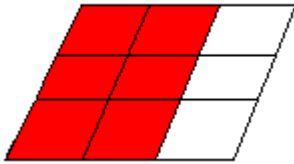
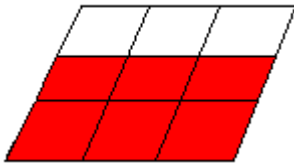
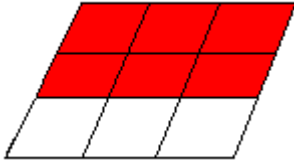
22 + 6 = 28

We can also make parallelograms from 4 parallelograms (2 adjacent and 2 directly below). There are 4 of these.



$$28 + 4 = 32$$

We can also make 2 "horizontal" parallelograms using 6 small parallelograms and 2 "vertical" ones as well.



And that's it. $32 + 4 = 36$ **Ans.**

11. 5 cm more than 3 times the length of a rectangle is less than or equal to 44 cm. It's also greater than or equal to 20 cm. The width of the same rectangle is 10 cm. We are asked to find the positive difference between the maximum possible area of the rectangle and the minimum possible area of the rectangle. Let l = the length of the rectangle. Let w = the width of the rectangle.

$$3l + 5 \leq 44$$

$$3l + 5 \geq 20$$

$$w = 10$$

Let's try and find the largest value of l .

$$3l + 5 \leq 44$$

$$3l \leq 39$$

$$l \leq 13$$

Therefore, 13 is the largest possible value for the length and the area is

$$13 \times 10 = 130$$

Now let's try and find the smallest value of l .

$$3l + 5 \geq 20$$

$$3l \geq 15$$

$$l \geq 5$$

5 is the smallest possible length of the rectangle and the smallest area is

$$5 \times 10 = 50$$

$$130 - 50 = 80$$
 Ans.

12. Javier needs to exchange a dollar bill for coins. The cashier has 2 quarters, 10 dimes and 10 nickels. We are asked to determine how many possible combinations of coins Javier could get back if he must get back at least one of the quarters.

Let's start by assuming he uses only 1 quarter.

D N

$$7 \quad 1$$

$$6 \quad 3$$

$$5 \quad 5 \quad \text{That's 5 combinations.}$$

$$4 \quad 7$$

$$3 \quad 9$$

Now, assume he's going to get both quarters.

D N

$$5 \quad 0$$

$$4 \quad 2$$

$$3 \quad 4 \quad \text{That's 6 combinations.}$$

$$2 \quad 6$$

$$1 \quad 8$$

$$0 \quad 10.$$

$$5 + 6 = 11$$
 Ans.

13. There are four test scores which have a median of 80 and a range of 12. We are asked to find the maximum possible score that could have been received on a test.

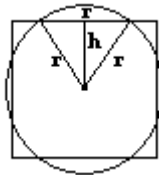
Since there is an even number of tests, either both middle tests were 80 or their sum was $2 \times 80 = 160$. The largest number we can choose for test #1 is 80 so that the range will be the maximum score for test #4. Test #2 and #3 would then be 80 each. Since the range is 12, then $80 + 12 = 92$ is the largest test score. **92 Ans.**

14. RS,TUV is a five-digit integer. It is divisible by 5, $S = R^2$, and $10 \times T + U = 5$. R, S, T, U and V are not necessarily distinct. We must find how many five-digit integers satisfy these conditions. If the five-digit integer is divisible by 5, then $V = 0$ or $V = 5$.

$10 \times T + U = 5$ must mean that $T = 0$ and $U = 5$. Otherwise the expression would be a value greater than 5. If $S = R^2$, then S could be 1 and R could be 1; or S could be 4 and R could be 2; or S could be 9 and R could be 3. So what does this all tell us? We can have: 11050, 11055, 42050, 42055, 93050, 93055 or 6 integers. 6 **Ans.**

15. $y = x^2 + 10x + 21$
 What is the least possible value of y ?
 First, let's find when $y = 0$.
 $x^2 + 10x + 21 = 0$
 $(x + 3)(x + 7) = 0$
 $x = -3$, and $x = -7$
 What happens when $x = -4$?
 $(-4)^2 + (10 \times -4) + 21 =$
 $16 - 40 + 21 = -3$
 What happens when $x = -5$?
 $(-5)^2 + (10 \times -5) + 21 =$
 $25 - 50 + 21 = -4$
 And what happens when $x = -6$?
 $(-6)^2 + (10 \times -6) + 21 =$
 $36 - 60 + 21 = -3$
 Since $x = -3$ and $x = -7$ give us expressions that evaluate to 0, the smallest value of the expression is -4 . **Ans.**

16. A square and a circle intersect so that each side of the square contains a chord of the circle which equals the circle's radius. We are asked to find the ratio of the area of the square to the area of the circle.
 The area of the circle is πr^2 . If we draw a line from the center of the circle to the circumference at the points where the chord is created, we have created an equilateral triangle of length r .



The height of that triangle is $\frac{1}{2}$ the length of the square.

$$h^2 + \left(\frac{r}{2}\right)^2 = h^2 + \frac{r^2}{4} = r^2$$

$$h^2 = \frac{3}{4}r^2$$

$$h = \frac{\sqrt{3}}{2}r$$

$$2h = \sqrt{3}r$$

The area of the square is $(\sqrt{3}r)^2 = 3r^2$

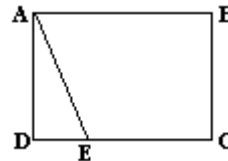
So the ratio of the area of the square to the area of the rectangle is

$$\frac{3r^2}{\pi r^2} = \frac{3}{\pi} \quad \text{Ans.}$$

17. In rectangle ABCD, point E is on side

CD. The area of triangle ADE is $\frac{1}{5}$ of

the area of quadrilateral ABCE. We are asked, what is the ratio of the length of segment DE to the length of segment DC?



Let $l = AD$ and let $w = DC$.

Then lw is the area of the rectangle.

Let $x = DE$.

Let $y =$ the area of quadrilateral ABCE.

$$\text{Then } \frac{1}{5}y + y = lw.$$

$$\frac{6}{5}y = lw \quad \text{and} \quad y = \frac{5}{6}lw$$

The area of triangle ADE is $\frac{xl}{2}$

$$\frac{xl}{2} = \frac{1}{5} \times \frac{5}{6}lw = \frac{1}{6}lw$$

$$3xl = lw \quad \text{and} \quad 3x = w$$

Therefore, $x = \frac{1}{3}w$ and w is DC.

$$\frac{1}{3} \quad \text{Ans.}$$

18. The sum of the first n numbers in a sequence of numbers is given by

$n^3 + 4n$ for all positive integers n . We are asked to find the tenth number in the sequence.

The first number is $1^3 + 4 = 5$. That is actually the first number.

The sum of the first 2 numbers is $8 + 8 = 16$. The second number is 11.

The sum of the first 3 numbers is $27 + 12 = 39$. The third number is 23.

The sum of the first numbers is $64 + 16 = 80$. The fourth number is 51.

So we have 5, 11, 23, 41. Can we see a pattern? I think so.

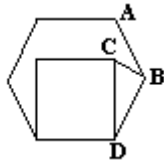
$$11 - 5 = 6$$

$$23 - 11 = 12$$

$$41 - 23 = 18$$

Therefore, the fifth number is $41 + 24 = 65$. The sixth is $65 + 30 = 95$. Seventh is $95 + 36 = 131$. Eighth is $131 + 42 = 173$. Ninth is $173 + 48 = 221$ and the tenth number is $221 + 54 = 275$. **Ans.**

19. We are given a square in the interior of a hexagon and asked to find the degree measure of $\angle ABC$.



Hexagons have interior angles of 120° . Let s = the side of the hexagon (and of the square).

Then $CD = s$ and $DB = s$ making CDB an isosceles triangle. Since $\angle D = 120^\circ$ and an angle of the square is 90° , $\angle BDC$ must be 30° . $\angle DCB = \angle DBC$
 $180 - 30 = 150$

Both $\angle DCB$ and $\angle DBC$ must then be 75° . Since $\angle B$ is also 120° ,
 $\angle ABC = 120 - 75 = 45$. **Ans.**

20. When the diameter of a pizza increased by 2 in, the area of the pizza increases by 44%. We are asked to find the area of the original pizza.

Let x = the radius of the original pizza. Then the area of the original pizza is πx^2 . Increasing the diameter by 2 in is the same as increasing the radius by 1 in.

$$\frac{\pi(x+1)^2}{\pi x^2} = 1.44$$

$$\frac{(x+1)^2}{x^2} = 1.44$$

$$\frac{x+1}{x} = 1.2$$

$$x+1 = 1.2x$$

$$0.2x = 1$$

$$2x = 10$$

$$x = 5$$

The area of the original pizza is 25π .

Ans.

21. When the numerator of a fraction is increased by six, the value of the fraction increases by 1. If the denominator of the fraction is increased by 36, the value of the fraction decreases by one.

Let $\frac{x}{y}$ be the fraction. Then

$$\frac{x+6}{y} = \frac{x}{y} + 1 \text{ and}$$

$$\frac{x}{y+36} = \frac{x}{y} - 1$$

$$\frac{x+6}{y} = \frac{x+y}{y}$$

$$x+6 = x+y$$

$$y = 6$$

Substituting into the second equation:

$$\frac{x}{6+36} = \frac{x}{6} - 1$$

$$\frac{x}{42} = \frac{x-6}{6}$$

$$6x = 42(x-6)$$

$$x = 7(x-6)$$

$$x = 7x - 42$$

$$6x = 42$$

$$x = 7$$

The original fraction was $\frac{7}{6}$. **Ans.**

22. The 80th term of an arithmetic sequence is twice the 30th term. If the first term of the sequence is 7, then what is the 40th term?

The second term is $7 + x$, the third is $7 + 2x$ and so on. That makes the 30th term $7 + 29x$. The 80th term is $7 + 79x$
 $7 + 79x = 2 \times (7 + 29x)$

$$7 + 79x = 14 + 58x$$

$$21x = 7 \text{ and } x = \frac{1}{3}$$

The 40th term is $7 + 39x$.

$$7 + 39x = 7 + \left(39 \times \frac{1}{3}\right) =$$

$$7 + 13 = 20 \text{ **Ans.**}$$

23. A right circular cone is sliced into four pieces which have the same height. We are asked to find the ratio of the volume of the second largest piece to the volume of the largest piece.

Let r = the radius of the cone and h = the height of the cone.

The radius grows proportionally as does the height so that for the smallest cone

the radius is $\frac{1}{4}r$ and the height is $\frac{1}{4}h$,

and for the cone made up of the first two pieces the radius is $\frac{1}{2}r$ and the height

is $\frac{1}{2}h$, and so on for the other two cones.

The second largest piece is the volume of the second largest cone (with radius $\frac{3}{4}r$ and height $\frac{3}{4}h$) minus the volume of the third largest cone (with radius $\frac{1}{2}r$ and height $\frac{1}{2}h$).

$$V_1 = \frac{1}{3} \times \pi \times \left(\frac{3}{4}r\right)^2 \left(\frac{3}{4}h\right) =$$

$$\frac{1}{3} \times \pi \times \frac{9}{16} \times \frac{3}{4}r^2h = \frac{9}{64}\pi r^2h$$

$$V_2 = \frac{1}{3} \times \pi \times \left(\frac{1}{2}r\right)^2 \left(\frac{1}{2}h\right) =$$

$$\frac{1}{3} \times \pi \times \frac{1}{4} \times \frac{1}{2}r^2h = \frac{1}{24}\pi r^2h$$

$$V_1 - V_2 = \frac{9}{64}\pi r^2h - \frac{1}{24}\pi r^2h =$$

$$\frac{27}{192}\pi r^2h - \frac{8}{192}\pi r^2h = \frac{19}{192}\pi r^2h$$

The volume of the largest piece is the volume of the entire cone minus the

volume of the cone with radius $\frac{3}{4}r$ and

height $\frac{3}{4}h$, or V_1 .

$$V_3 = \frac{1}{3}\pi r^2h$$

$$V_3 - V_1 = \frac{1}{3}\pi r^2h - \frac{9}{64}\pi r^2h =$$

$$\frac{64}{192}\pi r^2h - \frac{27}{192}\pi r^2h = \frac{37}{192}\pi r^2h$$

Taking the ratio:

$$\frac{\frac{19}{192}\pi r^2h}{\frac{37}{192}\pi r^2h} = \frac{19}{37} = \frac{19}{37} \text{ **Ans.**}$$

24. $x + y + z = 7$

$$x^2 + y^2 + z^2 = 19$$

We are asked to find the mean of the three products, xy , yz and xz .

$$(x + y + z)^2 = 7^2 = 49$$

$$(x + y + z)^2 =$$

$$x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2$$

$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz =$$

$$19 + 2(xy + xz + yz) = 49$$

$$2(xy + xz + yz) = 49 - 19 = 30$$

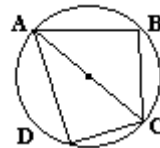
$$xy + xz + yz = 15$$

$$\frac{xy + xz + yz}{3} = \frac{15}{3} = 5 \text{ **Ans.**}$$

25. Quadrilateral ABCD is inscribed in a circle with segment AC a diameter of the circle. $\angle DAC = 30^\circ$ and $\angle BAC = 45^\circ$. The ratio of the area of ABCD to the area of the circle can be expressed as a

common fraction in the form $\frac{a + \sqrt{b}}{c\pi}$. We

are asked to find the value of $a + b + c$.



Because $\angle ADC$ and $\angle ABC$ each intercept a semi-circle, each one is a 90° angle. Therefore, $\triangle ACD$ is a 30-60-90 triangle and $\triangle ABC$ is a

45-45-90 triangle. The hypotenuse of each triangle is $2r$, where r is the radius of the circle. The two legs of $\triangle ACD$ are r and $r\sqrt{3}$ (using 30-60-90 properties)

and its area is $\frac{1}{2}r(r\sqrt{3}) = \frac{\sqrt{3}}{2}r^2$. The

legs of $\triangle ABC$ are each $r\sqrt{2}$ (using 45-45-90 properties) and its area is $\frac{1}{2}(r\sqrt{2})^2 = \frac{1}{2}(2r^2) = r^2$.

Therefore, the area of quadrilateral

ABCD is $\frac{\sqrt{3}}{2}r^2 + r^2 = \frac{2+\sqrt{3}}{2}r^2$. The

area of the circle is $\pi(r^2)$. So, the ratio of the area of the quadrilateral to the area of the circle is

$$\frac{\frac{2+\sqrt{3}}{2}r^2}{\pi r^2} = \frac{\frac{2+\sqrt{3}}{2}}{\pi} = \frac{2+\sqrt{3}}{2\pi}$$

This makes $a = 2$, $b = 3$, and $c = 2$.

$2 + 3 + 2 = 7$ **Ans.**

26. The numbers a , b , c and d form a geometric sequence.

$b = a + 3$ and $c = b + 9$. We are asked to find the value of d .

If $c = b + 9$, then $c = (a + 3) + 9 = a + 12$.

So we actually have the sequence:

$a, a + 3, a + 12, d$

In a geometric sequence, each term is n times the previous term.

$a + 3 = n \times a$

$$n = \frac{a+3}{a}$$

$$a + 12 = \frac{a+3}{a} \times (a+3)$$

$$a + 12 = \frac{a^2 + 6a + 9}{a}$$

$$a^2 + 12a = a^2 + 6a + 9$$

$$12a = 6a + 9$$

$$6a = 9$$

$$a = \frac{9}{6} = \frac{3}{2}$$

$$n = \frac{a+3}{a} = \frac{\frac{3}{2}+3}{\frac{3}{2}} = \frac{\frac{9}{2}}{\frac{3}{2}} = \frac{9}{3} = 3$$

Finally, now that we know what n is, let's determine the value of $a + 12$ and

multiply it by n .

$$a + 12 = \frac{3}{2} + \frac{24}{2} = \frac{27}{2}$$

$$d = n \times (a + 12) =$$

$$3 \times \frac{27}{2} = \frac{81}{2} = 40\frac{1}{2} \text{ **Ans.**}$$

27. If you flip an unfair coin three times, the probability of getting 3 heads is the same as the probability of getting exactly two tails. We are asked to find the ratio of the probability of flipping a tail to the probability of flipping a head. Let h = the probability of a head coming up.

Let t = the probability of a tail coming up. We have one possibility that gives us 3 heads, i.e., HHH and 3 possibilities of exactly two tails, TTH, THT, and HTT.

$$h^3 = 3 \times ht^2$$

$$h^2 = 3t^2$$

$$t^2 = \frac{h^2}{3}$$

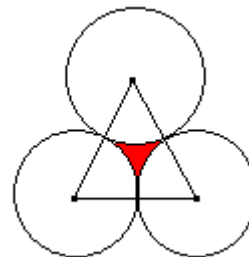
$$t = \sqrt{\frac{h^2}{3}} = \frac{h}{\sqrt{3}} = \frac{h\sqrt{3}}{3}$$

The ratio of the probability of flipping a tail to the probability of flipping a head is

$$\frac{t}{h} = \frac{\frac{h\sqrt{3}}{3}}{h} = \frac{\sqrt{3}}{3} \text{ **Ans.**}$$

28. The region shown is bounded by the arcs of circles having radius 4 units. Each arc has a central angle measure of 60° and intersects at points of tangency. The area of the region can be

expressed in the form $a\sqrt{b} + c\pi$. We are asked to find the value of $a + b + c$.



This turns out to be pretty simple. If you draw lines connecting the three

centers of the circles, you end up with an equilateral triangle (that's where the 60° comes in) whose sides are twice the radius or 8. We must first find the area of the triangle and for that we must have the length of the height of the triangle.

$$h^2 + 4^2 = 8^2$$

$$h^2 + 16 = 64$$

$$h^2 = 64 - 16 = 48$$

$$h = \sqrt{48} = 4\sqrt{3}$$

Therefore, the area of the triangle is:

$$\frac{1}{2} \times 8 \times 4\sqrt{3} = 16\sqrt{3}$$

From this, we need to subtract the area of the sections of the circle formed by the 60° central angles. But we have 3 of those which is equivalent to the area formed by a semicircle. The area of the circle is: $\pi r^2 = 16\pi$. Half of that is 8π .

So, the area of the region bounded by the arcs is $16\sqrt{3} - 8\pi$. This means:

$a = 16$, $b = 3$, $c = -8$
 $a + b + c = 16 + 3 - 8 = 11$ **Ans.**

29. A bag contains red and white balls. If 5 balls will be pulled from the bag with replacement, the probability of getting exactly 3 red balls is 32 times the probably of getting exactly one red ball. We are asked to find what percent of the balls originally in the bag are red.

Let r = the number of red balls.

Let w = the number of white balls.

Then the probability of picking a red ball

from the bag is $\frac{r}{r+w}$ and the

probability of picking a white ball from

the bag is $\frac{w}{r+w}$.

The probability of getting 3 red balls is:

$$\frac{r^3 w^2}{(r+w)^5}$$

There are $\frac{5!}{3!2!} = 10$ ways to do it.

The probability of getting only one red

ball is: $\frac{r w^4}{(r+w)^5}$

There are $\frac{5!}{4!1!} = 5$ ways to do it.

$$\frac{r^3 w^2}{(r+w)^5} \times 10 = 32 \frac{r w^4}{(r+w)^5} \times 5$$

$$r^3 w^2 = 16 r w^4$$

$$r^2 = 16 w^2$$

$$r = 4w$$

So, there are 4 more times as many red balls as white balls. That's 80%. **Ans.**

30. $52,683 \times 52,683 - 52,660 \times 52,706 = (52660 + 23)^2 - (52660 \times (52660 + 46)) = 52660^2 + 23^2 + (2 \times 23 \times 52660) - (52660^2 + (52660 \times 46)) = 23^2 + (46 \times 52660) - (46 \times 52660) = 23^2 = 529$ **Ans.**

TARGET ROUND

- We are asked to find the smallest possible integer such that none of the digits is a zero, the integer is a multiple of 3 and exactly one of the digits is a perfect square. A multiple of 3 means that if you add the digits in the number, the sum is itself a multiple of 3. The first perfect square is 1. We must have 2 other digits that are not perfect squares. That would have to be 2 and 3 so the smallest number is 123. **Ans.**
- We have a truck that is driven 480 miles at 60 miles per hour. The truck averages 20 miles per gallon. The driver is paid \$12/hour. Gasoline costs \$3 per gallon. Finally, maintenance on the truck is \$0.15 per mile. We must determine what the total cost of the trip is for the company. 480 miles at 60 miles per hour means that the truck is driven for $\frac{480}{60} = 8$ hours. So the labor cost is $12 \times 8 = 96$. 480 miles at 20 miles per gallon means we must buy $\frac{480}{20} = 24$ gallons of gas at \$3 per gallon. $24 \times 3 = 72$. $96 + 72 = \$168$, so far. Is there anything else? Yes, the maintenance.

$$480 \times 0.15 = 72$$

$$168 + 72 = 240 \quad \text{Ans.}$$

3. Jose had $\frac{1}{5}$ of the pizza. Marti had $\frac{1}{4}$ of what was left. Then Bobbi had $\frac{1}{3}$ of the remaining pizza and after that, Ms. Gauss took $\frac{1}{2}$ of that. So how much pizza is left?

After Jose had $\frac{1}{5}$ of the pizza there was

$$1 - \frac{1}{5} = \frac{4}{5} \text{ of the pizza left. Marti ate } \frac{1}{4}$$

of that or $\frac{1}{4} \times \frac{4}{5} = \frac{1}{5}$ of the pizza. Now

$$\frac{4}{5} - \frac{1}{5} = \frac{3}{5} \text{ of the pizza is left. Bobbi}$$

gets $\frac{1}{3}$ of that or $\frac{1}{3} \times \frac{3}{5} = \frac{1}{5}$ of the

pizza which leaves $\frac{3}{5} - \frac{1}{5} = \frac{2}{5}$ of the

pizza left. Finally Ms. Gauss gets her $\frac{1}{2}$ of the leftovers or $\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$ of the

pizza. That leaves $\frac{2}{5} - \frac{1}{5} = \frac{1}{5}$ of the

pizza left. $\frac{1}{5}$ or 20% **Ans.**

4. Marco will celebrate his Nth birthday in the year N^2 , which is in the 21st century. So, in what year did he celebrate his 13th birthday?
Okay, we have to find a square that is between 2000 and 2100.
 $40^2 = 1600$ and $50^2 = 2500$ We're looking for a value between 40 and 50, probably somewhere in the mid 40's range. $45^2 = 2025$ and that must be it!
So, Marco will be 45 in 2025.
 $45 - 13 = 32$ which means 32 years before 2025, Marco was 13.
 $2025 - 32 = 1993$ **Ans.**

5. How many distinct sums can be obtained by adding three different

numbers from the set $\{3, 6, 9, \dots, 27, 30\}$?

Let's rewrite that as

$\{3 \times 1, 3 \times 2, 3 \times 3, \dots, 3 \times 9, 3 \times 10\}$

And we can reduce this to how many values can you get adding three of the numbers 1 through 10.

There are $10 \times 9 \times 8 = 720$ different values. But many of them are the same.

The smallest value is $1 + 2 + 3 = 6$.

The largest value is $8 + 9 + 10 = 27$.

And we can get every value in between.

$27 - 6 = 21$ But don't forget to count the 6, because there are actually 22

numbers from 6 to 27. **22 Ans.**

6. x and y are real numbers such that $xy = 9$ and $x^2y + xy^2 + x + y = 100$

We are asked to find the value of

$$x^2 + y^2$$

$$x^2y + xy^2 + x + y = 100$$

$$x \times xy + y \times xy + x + y = 100$$

$$9x + 9y + x + y = 100$$

$$10(x + y) = 100$$

$$x + y = 10$$

$$(x + y)^2 = 100 = x^2 + 2xy + y^2$$

$$x^2 + y^2 + (2 \times 9) = 100$$

$$x^2 + y^2 = 100 - 18 = 82 \quad \text{Ans.}$$

7. In a shooting competition, the object of the match is to be the first to hit the bull's eye of a target 100 feet away. Each opponent has a 40% chance of hitting the bull's-eye on a given shot. Franz shoots first and we must determine the probability that Hans will win the competition and take no more than 3 shots.

Scenario 1: Franz misses and then Hans hits the bull's-eye.

The probability of this is

$$0.6 \times 0.4 = 0.24$$

Scenario 2: Franz misses, Hans misses, Franz misses and Hans hits the bull's-eye.

$$0.6 \times 0.6 \times 0.6 \times 0.4 = .0864$$

Scenario 3: Franz misses, Hans misses, Franz misses, Hans misses, Franz misses and Hans hits the bull's-eye.

$$0.6 \times 0.6 \times 0.6 \times 0.6 \times 0.6 \times 0.4 =$$

$$0.031104$$

The probability of all 3 scenarios is:

$$0.24 + 0.0864 + 0.031104 =$$

$$0.357504 \approx 0.36 \quad \text{Ans.}$$

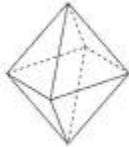
8. The solid figure has 6 faces that are squares and 8 faces that are equilateral triangles. Each of the 24 edges has length 2 cm. The volume of the solid can be expressed in the form

$$\frac{a}{b}\sqrt{c}$$

where c has no perfect square factor except 1, and a and b are relatively prime. We must find the value of $a + b + c$.

OK. I'm not going to even try to reproduce that figure so I'm just going to talk through it. First of all, there is a rectangular solid in there whose length and width are each 2 cm and whose height is the diagonal of a square whose side is 2 cm. The diagonal must be

$2\sqrt{2}$ and the volume of the solid is $2 \times 2 \times 2\sqrt{2} = 8\sqrt{2}$. Now, if you remove that part of the object, you're left with 4 half-octahedrons. An octahedron looks like



this:

We actually, then, have 2 full octahedrons. The volume of an octahedron with side 2 is

$$\frac{1}{3} \times \sqrt{2} \times 2^3 = \frac{8}{3}\sqrt{2}$$

The volume of two of them is $\frac{16}{3}\sqrt{2}$.

$$8\sqrt{2} + \frac{16}{3}\sqrt{2} = \frac{40}{3}\sqrt{2}$$

This means that $a = 40$, $b = 3$ and $c = 2$. $a + b + c = 40 + 3 + 2 = 45$ **Ans.**

TEAM ROUND

1. Kevin, Cindi and Marcus have 1020 widgets. Marcus has half of what Cindi has and Kevin has 219 widgets. So how many widgets does Cindi have? Let x = the number of widgets that Marcus has. Then Cindi has $2x$ widgets.

$$1020 - 219 = 801$$

$$x + 2x = 801$$

$$3x = 801$$

$$x = 267$$

$$2x = 534$$
 Ans.

2. We have a three-digit integer where all digits are prime and distinct. They also increase in order from left to right. We are asked to determine how many possibilities there are for the number. Let's list all one-digit primes. 2, 3, 5, 7. The possibilities are: 235, 237, 257, 357 for a total of 4 **Ans.**
3. Every 25th customer receives a free Frisbee. Every 35th customer receives a free baseball hat. How many customers were there before someone gets both. Uh, we just have to find the LCM of 25 and 35 and then remember to subtract 1. $25 = 5 \times 5$
 $35 = 5 \times 7$
The LCM = $5 \times 5 \times 7 = 25 \times 7 = 175$. $175 - 1 = 174$ customers who came before the customer who received both. 174 **Ans.**

4. A train travels at a constant rate of 55 mph through a tunnel. 45 seconds after the front of the train enters the tunnel, the front of the train exits the tunnel. So how many feet long is the tunnel? How far can you go at 55 mph in 45 seconds.

$$55 \text{ mph} = \frac{55}{60} = \frac{11}{12} \text{ miles per minute.}$$

$$\frac{11}{12} \times \frac{45}{60} = \frac{11}{12} \times \frac{3}{4} = \frac{11}{4} \times \frac{1}{4} = \frac{11}{16} \text{ miles per 45 seconds.}$$

$$\frac{11}{16} \times 5280 = 330 \times 11 = 3630 \text{ ft}$$
 Ans.

5. An L-shaped piece is placed on the grid so that it covers three of the unit squares of the grid. The sum of the covered squares is S . We are allowed to rotate the L-shaped piece. So we need to find the values of S for all possible placements on this grid.

1	2	3
4	5	6
7	8	9

Let's look at the numbers 1, 2, 4, and 5. We can place the L-shaped piece on {1,

2, 5}, {2, 5, 4}, {1, 4, 5} and {1, 2, 4}.

The sums are 8, 11, 10 and 7.

$$8 + 11 + 10 + 7 = 36$$

Now, look at 2, 3, 5, 6. The sums are 11, 14, 13, and 10.

$$11 + 14 + 13 + 10 = 48$$

Now, look at 4, 5, 7, 8. The sums are 16, 20, 19, and 17.

$$16 + 20 + 19 + 17 = 72$$

Finally, look at 5, 6, 8, 9. The sums are 19, 23, 22, 20.

$$19 + 23 + 22 + 20 = 84$$

$$36 + 48 + 72 + 84 = 240 \text{ Ans.}$$

6. We have a container with four colors of marbles, red, yellow, blue and green. All but 45 of the marbles are red. All but 45 are yellow. All but 45 are blue. All but 60 are green. We must determine how many marbles are green.

Let r = the number of red marbles.

Let y = the number of yellow marbles.

Let b = the number of blue marbles.

Let g = the number of green marbles.

Let t = the total number of marbles.

$$t = r + 45$$

$$t = y + 45$$

$$t = b + 45$$

$$t = g + 60$$

$$\text{If } r + 45 = y + 45 = b + 45, \text{ then}$$

$$r = y = b.$$

From $t = g + 60$, we surmise that

$$r + y + b = 60.$$

$$3r = 60$$

$$r = y = b = 20.$$

$$t = r + 45 = 20 + 45 = 65$$

$$\text{Therefore, } g = 65 - 60 = 5. \text{ Ans.}$$

7. If 3 cookies are placed in a bag, then there are 2 cookies left over. If 5 cookies are placed in a bag, there are no cookies left over. If 8 cookies are placed in a bag, then there are 6 cookies left over. Determine the fewest number of cookies that could have been baked. Obviously, the number of cookies is a multiple of 5. One more than the number of cookies baked is divisible by 3 (because there are 2 cookies left over in the first scenario. The number of cookies must at least 15. (One bag of 8 cookies plus 6 left over and divisible by 5.) Find the first number divisible by 3 that is one more than a multiple of 5.

That's 21 so if we baked 20 cookies and put them into bags of 8, 4 would be left over. Not there yet. What's next?

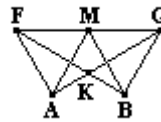
36 is the next candidate. If we baked 35 cookies and put them in bags of 8 – wait, we better be baking an even number of cookies. That means skipping every other possibility.

Next is 51. If we baked 50 cookies and put them in bags of 8, 2 would be left over.

Next is 81. If we baked 80 cookies and put them in bags of 8 there would be 0 left over.

It's got to get 111. If we baked 110 cookies and put them in bags of 8 there would be 6 left over. 110 Ans.

8. M is the midpoint of segment FG. A and B are points coplanar to points F and G. A and B are positioned on the same side of the line containing segment FG such that triangles FMA and MGB are equilateral. The lines FB and GA intersect at point K. So what is the measure of $\angle GKB$? Okay. We have to draw this out.



Triangle FMA and triangle MGB are equilateral. Therefore, $\angle FMA$ and $\angle GMB$ are both 60° . That means that $\angle AMB$ is $180^\circ - 60^\circ - 60^\circ = 60^\circ$. M and K are opposite points in a quadrilateral whose other 2 points are the intersections of lines FB and MA, and lines GA and MB, respectively. The sum of two opposite angles in a quadrilateral is 180° . Therefore, $\angle FKG$ is 120° . $\angle GKB$ is just the supplement, or 60° . Ans.

9. A bag contains 5 red marbles, 3 blue marbles and 2 green marbles. We draw 6 marbles from the bag with replacement. We must find the probability that two marbles of each color will be drawn.

There are a total of $5 + 3 + 2 = 10$ marbles in the bag. With all 10 in the

bag, there is a $\frac{5}{10} = \frac{1}{2}$ chance that we

draw a red marble. There is a $\frac{3}{10}$ chance that we draw one blue

marble. There is a $\frac{2}{10} = \frac{1}{5}$ chance that

we draw 1 green marble.

$$\left(\frac{1}{2}\right)^2 \times \left(\frac{3}{10}\right)^2 \times \left(\frac{1}{5}\right)^2 =$$

$$\frac{1}{4} \times \frac{9}{100} \times \frac{1}{25} = \frac{9}{10000}$$

is the probability that we draw two of each.

There are

$$\frac{6!}{2!2!2!} = \frac{6 \times 5 \times 4 \times 3}{4} = 90 \text{ possibilities.}$$

$$\frac{9}{10000} \times 90 = \frac{810}{10000} = \frac{81}{1000} \text{ Ans.}$$

10. The geometric mean of two positive numbers a and b is \sqrt{ab} . The third term of an arithmetic sequence of positive numbers is the geometric mean of the first and eleventh terms. We must find the ratio of the second term to the first term of the sequence.

In an arithmetic sequence, the terms differ by some value n .

If x is the first term, then $x + n$ is the second, $x + 2n$ is the third, ..., and $x + 10n$ is the 11th term.

$$x + 2n = \sqrt{x \times (x + 10n)}$$

$$x + 2n = \sqrt{x^2 + 10nx}$$

$$(x + 2n)^2 = x^2 + 4nx + 4n^2 = x^2 + 10nx$$

$$4n^2 = 6nx$$

$$4n = 6x$$

$$n = \frac{6}{4}x = \frac{3}{2}x$$

The ratio of the second term in the sequence to the first term in the sequence is

$$\frac{x + n}{x} = \frac{x + \frac{3}{2}x}{x} = \frac{\frac{5}{2}x}{x} = \frac{5}{2} \text{ Ans.}$$